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LETTER TO THE EDITOR

# Exact solution of Potts random energy models with weak connectivity and of the many-states Potts spin glass

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**Abstract.** We generalise Derrida's random energy model to the Potts spin glass with $p$-spin interactions in the limit of large $p$. We solve the model exactly for the cases of strong and weak connectivity. In the latter case we show how to solve the model using replicas and obtain a solution for the value of the set of order parameters $Q^{r_1 \dots r_s}_{\alpha_1 \dots \alpha_s}$. We then derive the large-$q$ limit of the ordinary $q$-state Potts spin glass on a lattice with finite connectivity.

Much interest has been devoted recently to spin glasses and other frustrated models defined on random lattices with diluted bonds, where the bond probability distribution is of the form

$$\mathscr{P}(J_{ij}) = (1-\gamma)\delta(J_{ij}) + \gamma\rho(J_{ij}). \qquad (1)$$

Such models arise naturally in the theory of combinatorial optimisation, like the graph partitioning problem [1–3]. Two cases may be distinguished.

(i) $\gamma$ is of order unity and $\langle J_{ij}^2\rangle - \langle J_{ij}\rangle^2 \sim 1/N$ where $N$ is the number of lattice sites. This is the strong-connectivity case which gives rise to an infinite-ranged model of the Sherrington–Kirkpatrick (SK) type [4].

(ii) On the other hand, when $J_{ij}$ are of order unity but $\gamma = c/N$ one obtains a model with an average finite connectivity. This model may still be thought of as some kind of mean-field theory because the probability of small loops is of $O(1/N)$, so locally the lattice looks like a tree [2]. This mean-field theory differs from the infinite-ranged model which corresponds to case (i).

It has been shown recently [5–7] that for the Ising and Potts models defined on lattices with weak connectivity, replica symmetry breaking (RSB) occurs in the vicinity of the critical point, and numerical evidence [8] indicates that this phenomenon continues to low temperatures. Recently De Dominicis and Mottishaw [9] found a solution to the Derrida $p$-spin model [10, 11] when $p \to \infty$ in the weak-connectivity case at any temperature in the presence of RSB.

Our aim here is twofold. First we show how to extend the $p$-spin model to the case of Potts variables, which has not been done previously, and obtain the solution of the model in the strong- and weak-connectivity cases when $p \to \infty$ for any value of $q$, the number of Potts states. We also show that when $q \to \infty$ the limit of large $p$ is

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redundant; hence the solution of the many-states ordinary Potts spin glass in the weak-connectivity case is obtained.

We have found that in order to define a generalisation of the Derrida $p$-spin model one cannot start from the ordinary Potts glass Hamiltonian, but rather from the 'gauge invariant' Potts spin-glass model [12, 13]. We define the Potts $p$-spin model by the classical Hamiltonian

$$\mathscr{H} = \sum_{(i_1,\ldots,i_p)} \sum_{r=1}^{q-1} J^{(r)}_{i_1,\ldots,i_p} \sigma^r_{i_1} \ldots \sigma^r_{i_p} \tag{2}$$

where the variables $\sigma_i$ take their values among the $q$th-order roots of unity and the sum $(i_1, \ldots, i_p)$ is over distinct $p$-plets of spins. The couplings $J^{(r)}_{i_1,\ldots,i_p}$ are quenched random variables which are distributed according to a generalisation of (1). One can choose $\rho$ to be a Gaussian distribution, but instead in this letter we use a discrete distribution

$$J^{(r)}_{i_1,\ldots,i_p} = J\tau^r_{i_1,\ldots,i_p} \tag{3}$$

with

$$\mathscr{P}(\tau_{i_1,\ldots,i_p}) = (1-\gamma)\delta(\tau_{i_1,\ldots,i_p}) + \gamma\frac{1}{q}\sum_{l=0}^{q-1} \delta(\tau_{i_i,\ldots,i_p} - \theta^l) \tag{4}$$

and

$$\theta = \exp(2\pi \mathrm{i} q). \tag{5}$$

Substituting (3) into (2), the Hamiltonian can be written in the form

$$\mathscr{H} = -J \sum_{(i_1,\ldots,i_p)} (q\delta_{\tau_{i_1,\ldots,i_p}\sigma_{i_1,\ldots,}\sigma_{i_p},1} - 1). \tag{6}$$

This Hamiltonian possesses the local gauge symmetry

$$\sigma_i \to \mu_i\sigma_i \qquad \tau_{i_1,\ldots,i_p} \to \mu^*_{i_1}, \ldots, \mu^*_{i_p}\tau_{i_1,\ldots,i_p} \tag{7}$$

and the same is true for the probability distribution $\mathscr{P}(\tau_{i_1,\ldots,i_p})$, where $\mu_i$ are chosen from among the set of $q$th roots of unity.

For the case $\gamma = 1$ and $J^2 = J_0^2 p!/N^{p-1}$, this model becomes a random energy model when $p$ tends to infinity. In this case it can be readily verified that the probability of $M$ arbitrary, macroscopically distinct configurations to have energies $E_1, \ldots, E_M$ factorises in the limit $p \to \infty$ (in an analogous calculation to [10]):

$$P(E_1, \ldots, E_M) = \prod_{i=1}^{M} P(E_i) \tag{8}$$

$$P(E) = (\pi J_0^2 N(q-1))^{-1/2} \exp\{-E^2/[J_0^2 N(q-1]\}. \tag{9}$$

This model has $q^N$ states instead of Derrida's $2^N$, and its free energy site for $T > T_c$ is given by ($\beta = 1/T$):

$$-\beta f = \ln q + \tfrac{1}{4}\beta^2 J_0^2 (q-1) \tag{10}$$

and for $T < T_c$ the free energy is frozen and given by

$$f = -J_0[(q-1)\ln q]^{1/2} \tag{11}$$

where

$$T_c = \tfrac{1}{2} J_0[(q-1)/\ln q]^{1/2} \tag{12}$$

is the temperature below which the entropy vanishes. Of course one could define the $q^N$-state random energy model as a model of $q^N$ independent random energy levels distributed according to (7) but the model (2) provides a useful realisation which can later be extended to the weak-connectivity case. It also allows an investigation of the spin-glass order parameters using replicas.

We make here two remarks: firstly that in the strong-connectivity case the results are independent of the particular distribution chosen; secondly, the ordinary Potts spin glass defined by the Hamiltonian

$$\mathcal{H} = -q \sum_{i<j} J_{ij} \delta_{\sigma_i, \sigma_j} \tag{13}$$

gives, in the limit of large $q$ [14], results similar to those represented by equations (10)-(12). Better understanding of this point will be gained below.

We now turn to the weak-connectivity case for which we take $\gamma = cp!/2N^{p-1}$ in equation (4) and $J = J_0$ is of O(1) with respect to $N$. In that case we evaluate the probability of $M$ distinct configurations to have energies $E_1, \ldots, E_M$ and find in the limit of large $p$ (analogously to [9]):

$$P(E_1, \ldots, E_M) = \int \frac{d\hat{x}}{2\pi} Q(\hat{x}) \prod_{l=1}^{M} P(E_l, \hat{x}) \tag{14}$$

where

$$Q(\hat{x}) = \int dx \exp\{ix\hat{x} + \tfrac{1}{2}cN[\exp(x) - 1]\} \tag{15}$$

$$P(E, \hat{x}) = \int \frac{d\omega}{2\pi} \exp(i\omega E) \exp\left[-i\hat{x} \ln\left(\frac{1}{q} \exp[i\omega J_0(q-1)] + \frac{q-1}{q} \exp(-i\omega J_0)\right)\right]. \tag{16}$$

From (14)-(16) we derive the free energy per site

$$-\beta f = \ln q + \tfrac{1}{2}c \ln\left(\frac{1}{q} \exp[\beta J_0(q-1)] + \frac{q-1}{q} \exp(-\beta J_0)\right) \tag{17}$$

for $T > T_c$. The critical temperature $T_c$ is the temperature at which the entropy vanishes. Below $T_c$ the entropy is null and the free energy remains frozen.

This happens only for $c > 2$. When $c < 2$

$$S(T=0) = (1 - c/2) \ln q \tag{18}$$

and there is no spin-glass transition. Note that if $J_0 \to J_0/\sqrt{c}$ and $c \to \infty$ one recovers (10) from (17). Also the result (17) does depend on the particular bond distribution taken. We now derive this result using the replica approach. Furthermore, we show that in the large-$q$ limit, the result stands even without taking the $p \to \infty$ limit, i.e. it is valid for $p = 2$ as has been true in the strong-connectivity case. For the case $q = 2$, (17) agrees with the result of [9].

We use the following identity which we derived previously [7]:

$$\exp\left(\beta J_0 \sum_{\alpha=1}^{n} I(\sigma_\alpha, 1)\right) = \sum_{s=0}^{\infty} b_s \sum_{(\alpha_1, \ldots, \alpha_s)} I(\sigma_{\alpha_1}, 1) \tag{19}$$

with

$$I(\sigma, 1) = q\delta_{\sigma,1} - 1 \tag{20}$$

$$b_s = A^n \mu^s \qquad A = \frac{1}{q}\exp[\beta J_0(q-1)] + \frac{q-1}{q}\exp(-\beta J_0) \qquad \mu = \frac{\exp(q\beta J_0)-1}{\exp(q\beta J_0)+q-1} \tag{21}$$

to express the free energy in the form

$$-\beta f n \equiv -\sum_{s=2}\sum_{(\alpha_1,\ldots,\alpha_s)}\sum_{r_1,\ldots,r_s}{}' Q^{r_1,\ldots,r_s}_{\alpha_1,\ldots,\alpha_s} \mathrm{i}\lambda^{r_1,\ldots,r_s}_{\alpha_1,\ldots,\alpha_s} + \tfrac{1}{2}c\sum_{s=2} b_s \sum_{(\alpha_1,\ldots,\alpha_s)}\sum_{r_1,\ldots,r_s}{}' (Q^{r_1,\ldots,r_s}_{\alpha_1,\ldots,\alpha_s})^p$$
$$+\ln\tilde{Z} + \tfrac{1}{2}cn\ln\left(\frac{1}{q}\exp[\beta J_0(q-1)] + \frac{q-1}{q}\exp(-\beta J_0)\right) \tag{22}$$

$$\tilde{Z} = \operatorname*{Tr}_{\sigma_\alpha}\exp\left(\sum_{s=2}\sum_{(\alpha_1,\ldots,\alpha_s)}\sum_{r_1,\ldots,r_s}{}' \mathrm{i}\lambda^{r_1,\ldots,r_s}_{\alpha_1,\ldots,\alpha_s}\sigma^{r_1}_{\alpha_1},\ldots,\sigma^{r_s}_{\alpha_s}\right) \tag{23}$$

where the sum over $r_i$ runs from 1 to $q-1$ and the prime stands for the constraint $r_1+\ldots+r_s=0 \pmod q$, and the sum over replicas is over distinct sets.

We now introduce a one-step RSB by dividing the replicas into $(n/m)$ boxes, each of size $m$: $\alpha=(K,\gamma)$ where $K$ is the box number and $\gamma$ is the replica number in a box. We then parametrise $Q$ (and $\lambda$) in the following way:

$$Q^{r_1,\ldots,r_s}_{\alpha_1,\ldots,\alpha_s} = Q^{(s)}_{\{\nu_t,\eta_t\}}. \tag{24}$$

Here $\nu_t$ is the number of boxes of replicas with $t$ spins in a box for which the sum of the corresponding upper indices satisfies $r_{j_1}+\ldots+r_{j_t}=0 \pmod q$, whereas $\eta_t$ is the number of boxes with $t$ spins for which the upper indices satisfy $r_{j_1}+\ldots+r_{j_t}\neq 0 \pmod q$. Of course

$$\sum_{t=1}^{s} t(\nu_t+\eta_t) = s. \tag{25}$$

Similarly one defines

$$\Gamma^{(s)}_{\{\nu_t,\eta_t\}} = \sum_{\{K_1\gamma_1,\ldots,K_s\gamma_s\}}\sum_{r_1,\ldots,r_s}{}' \sigma^{r_1}_{K_1\gamma_1},\ldots,\sigma^{r_s}_{K_s\gamma_s}\Big|_{\{\nu_t,\eta_t\}} \tag{26}$$

where the sum is restricted to a given partition of the spins into the boxes and the sum of $r$ variables in each box is constrained as given by the variables $\{\nu_t,\eta_t\}$ (i.e. $\delta_{\Sigma r,0}$ for $\nu$ and $(1-\delta_{\Sigma r,0})$ for $\eta$). The function $\tilde{Z}$ in (23) can thus be written as

$$\tilde{Z} = \operatorname*{Tr}_{\sigma^\alpha}\exp\left(\sum_{s=2}\sum_{\{\nu_t,\eta_t\}} \mathrm{i}\lambda^{(s)}_{\{\nu_t,\eta_t\}}\Gamma_{\{\nu_t,\eta_t\}}(\{T_{K,j}\})\right) \tag{27}$$

where we introduce the variables

$$T_{K,j} = \sum_{l=0}^{q-1}\sum_{\gamma=1}^{m}\theta^{-jl}\sigma^l_{K\gamma} \tag{28}$$

and $\theta$ has been defined in (5), $K=1,\ldots,(n/m)$ and $j=0,\ldots,q-1$. The crucial point is that we have shown that in the large-$p$ limit all the spins in a given box lock into a common value. This means that only such configurations dominate the sum in (27).

The trace over the spins is effectively replaced by the measure

$$\prod_K\sum_{l=0}^{q-1}\prod_\gamma \delta(\sigma_{K\gamma}-\theta^l) \tag{29}$$

or, alternatively, for the variables $T_{K,j}$ by the measure

$$P(\{T_{K,j}\})=\prod_K\left[\sum_{l=0}^{q-1}\left(\delta(T_{K,l}-mq)\prod_{i\neq l}\delta(T_{K,i})\right)\right]. \tag{30}$$

It is then straightforward to show that in (27) only those $\Gamma^{(s)}$ which are labelled only by the variables $\nu_t$ survive (i.e. any $\Gamma^{(s)}$ for which any $\eta_t\neq 0$ does not contribute to the sum) and

$$\ln\tilde{Z}=\frac{n}{m}\ln q+\sum_{s=2}\sum_{\{\nu_t\}}\mathrm{i}\lambda^{(s)}_{\{\nu_t\}}\Gamma^{(s)}_{\{\nu_t\}}(\{\sigma_{K\gamma}=1\}). \tag{31}$$

Since the first term on the RHS of (22) can be written as

$$-\sum_s\sum_{\{\nu_t,\eta_t\}}Q^{(s)}_{\{\nu_t,\eta_t\}}\mathrm{i}\lambda^{(s)}_{\{\nu_t,\eta_t\}}(\{\sigma_{K\gamma}=1\}) \tag{32}$$

it follows by the stationarity condition with respect to $\lambda$, that below $T_c$

$$Q^{(s)}_{\{\nu_t\}}=1 \tag{33}$$

and all the $Q$ which are labelled by some $\eta_t\neq 0$ vanish. Hence we find

$$\begin{aligned}\sum_s b_s\sum_{(\alpha_1,\dots,\alpha_s)}\sum_{r_1,\dots,r_s}{}'(Q^{r_1,\dots,r_s}_{\alpha_1,\dots,\alpha_s})^p&=\sum_{s=2}\mu^s\sum_{\{\nu_t\}}\Gamma^{(s)}_{\{\nu_t\}}(\{\sigma_{K\gamma}=1\})\\&=\exp\left[\frac{n}{m}\ln\left(\sum_{s=0}^{\infty}C^s_m\mu^s\frac{[(q-1)^s+(-1)^s(q-1)]}{q}\right)\right]-1.\end{aligned} \tag{34}$$

The last expression can be easily evaluated, and we finally find ($T<T_c$):

$$-f=\frac{1}{\beta m}\ln q+\frac{1}{\beta m}\ln\left(\frac{1}{q}\exp[\beta mJ_0(q-1)]+\frac{q-1}{q}\exp(-\beta mJ_0)\right). \tag{35}$$

Stationarity with respect to $m$ requires $m=\beta_c/\beta$ where $\beta_c$ is the value for which the entropy vanishes. Thus we reover the exact result stated below (17).

We now turn to the large-$q$ limit of the ordinary Potts spin glass. In the case $p=2$ equation (22) takes the form

$$\begin{aligned}-\beta fn=&-\tfrac{1}{2}c\sum_{s=2}b_s\sum_{(\alpha_1,\dots,\alpha_s)}\sum_{r_1,\dots,r_s}{}'(Q^{r_1,\dots,r_s}_{\alpha_1,\dots,\alpha_s})^2\\&-\ln\operatorname*{Tr}_{\sigma_\alpha}\exp\left(\sum_{s=2}b_s\sum_{\{\alpha_1,\dots,\alpha_s\}}\sum_{r_1,\dots,r_s}{}'Q^{r_1,\dots,r_s}_{\alpha_1,\dots,\alpha_s}\sigma^{r_1}_{\alpha_1},\dots,\sigma^{r_s}_{\alpha_s}\right)\\&+n\tfrac{1}{2}c\ln\left(\frac{1}{q}\exp[\beta J_0(q-1)]-\frac{q-1}{q}\exp(-\beta J_0)\right).\end{aligned} \tag{36}$$

Notice that the second term in (34) is identical to (the logarithm of) (29) with $\lambda$ replaced by $Q$.

It turns out that for $q\to\infty$ and low temperature, the locking of the spin variables in the same box occurs, and one can use (30) and its consequences; in particular (33) follows by stationarity with respect to $Q$ and all other $Q$ with some $\eta_t\neq 0$ being null. One can then proceed similarly to (34). It turns out that when $q\to\infty$ one has to scale the temperature in the following way:

$$\exp(q\beta J_0)=1+q\tilde{K} \tag{37}$$

where $\tilde{K}$ is of O(1) with respect to $q$. The critical point is then given by

$$\tilde{K}_c = 2/(c-2) + O(1/\ln q). \tag{38}$$

The free energy is given by the large-$q$ limit of (17) and (35):

$$-\beta f = \ln q + \tfrac{1}{2}c \ln(1+\tilde{K}) + O(1/q) \qquad \tilde{K} < \tilde{K}_c \tag{39}$$

$$-\beta_c f = \ln q + \tfrac{1}{2}c \ln(1+\tilde{K}_c) + O(1/q) \qquad \tilde{K} > \tilde{K}_c. \tag{40}$$

Below $T_c$ the entropy vanishes and

$$N^{-1}E = -J_0 q[1 + O(1/\ln q)]. \tag{41}$$

It is interesting to note that below $T_c$ the energy is independent of $c$ to leading order.

The solution of the ordinary Potts spin glass on the weak-connectivity lattice in the limit of large $q$ is important, since if one could calculate corrections to the leading behaviour one would be able to understand the structure of the finite-$q$ models which are not yet understood away from the vicinity of $T_c$. The solution can also be generalised to include a magnetic field.

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